Two approaches for Helmholtz equation: generalized Darboux Transformation and the method of $\bar{\partial}$-problem

E.Sh.Gutshabash

Institute of Physics, St. -Petersburg State University, Russia
e-mail: gutshab@EG2097.spb.edu

Abstract

Two approaches to solution of the two-dimensional Helmholtz equation with a "wave number" are proposed. The results can be applied both in numerical areas of physics and in the theory of nonlinear equations. The first approach is based on the requirement of the covariance of equation under the generalized Darboux transformation (Moutard transformation). It allows to construct a new solution of equation, using a given initial solution of the equation. Simultaneously we obtain the "dressing" relation for the "wave number". The simplest examples of the approach are considered in detail. In the second approach the Green-Cauchy formula (the $\bar{\partial}$-method) is applied to reduce the solution of the equation to the solution of a system of singular integral equations.

1. INTRODUCTION

An intensive development of the methods in the theory of nonlinear integrable equa stimulated as well the development of the so-called dressing methods for the solution of a of linear equations having the important physical applications. One of the most popula effective methods is the method of Darboux Transformation (DT) [1], which gives the solutions of different one-dimensional equations. The situation for the multidimensional tions is more complicate. In [2] the multidimensional variant of DT, which depends on s derivatives, was considered, but such approach is rather complicate and cumbersome.

In paper [3] the method of integration of the non-stationary Schrodinger and Fokker- equations was proposed with using of the Darboux-like anzats and introducing of some funct differential equation. It appeared, that in some sense this approach is similar to the so- Moutard transformation (see, for example, [4]) for the variable real "wave number" (pote In this case the functional-differential equation of the second order is replaced by the syst two differential equations of the first order for functional, included in Darboux's anzats. F complex "wave number" the situation is more complicate - the solution can be found in simple cases only.

In Section 2 of this paper the idea of a Darboux-type anzats - Moutard transformatio introduced in the example of the two-dimensional Helmholtz equation. The main advanta such approach is independence of the procedure on the spatial derivatives, and hence it c straightforwardly extended onto higher dimensionality of space.

In Section 3 the simplest classical problem of diffraction of the electromagnetic wav a half-plane is considered. We reformulate this problem in terms of a so-called meth $\bar{\partial}$-problem. The idea of such approach (in another formulation of the problem) belo V.D.Lipovskii [5]. The powerful and effective possibilities of this method for integration o tidimensional systems such as Kadomtsev-Petviashvili-2 equation, Devey-Stewartson-2 s and so on were demonstrated (see, for example, [6]-[7]). Apparantely, up to now this app was not applied in the theory of diffraction. Similar to the Wiener-Hopf method, it allo formulate the initial boundary problem in terms of the integral equations, which admit t

2. The generalized Darboux Transformation

a). The two-dimensional Helmholtz's equation belongs to a wide class of the equatio
the following form (see, for example, [8]):

$$\Psi_{pq} = V(p,q)\Psi,$$

where $V = V(p,q)$ is the given, in general case, complex-valued function. Here and belo
use the notations: $\Psi_q = \partial\Psi/\partial q, \ldots$.

In the simplest particular case: $p = x,\ q = y,\ V(x,y) \equiv 1$, this equation has the ge
solution

$$\Psi(x,y) = \int_0^x f_1(s)J_0(2i\sqrt{y(x-s)}\,)\,ds + \int_0^y f_2(s)J_0(2i\sqrt{x(y-s)}\,)ds+$$

$$+[f_1(0) + f_2(0)]J_0(2i\sqrt{xy}\,),$$

where $J_0(.)$ is the Bessel's function, and $f_1,\ f_2$ are some differentiable functions.

The class of such equations is important both for the differential geometry and for a
variety of the physical problems. Indeed, for example, for $p = x + t,\ q = x - t$ we hav
heterogeneous wave equation:

$$\Psi_{tt} - \Psi_{xx} = V(x,t)\Psi,$$

and for $p = z = x + iy,\ q = \bar{p}$ we obtain the Helmholtz equation itself:

$$\triangle\Psi = V(x,y)\Psi.$$

We shall use it also in the complex form $(\partial_z = (1/2)(\partial_x - i\partial_y),\ \partial_{\bar z} = (1/2)(\partial_x + i\partial_y))$

$$\Psi_{z\bar z} = \frac{1}{4}V(z,\bar z)\Psi.$$

The equation (4(5)) arises in the Quantum Mechanics (two-dimensional stationary Schrö
equation), theory of diffraction of electromagnetic waves, an acoustic diffraction and other
lems. From the nonlinear equations point of view the interest to (4) is caused by his r
the associated linear system for the well-known nonlinear completely integrable Novikov-Ve
equation [9].

It is not difficult to check by direct calculation that if $\Phi = \Phi(z,\ \bar z),\ \chi = \chi(z,\ \bar z)$ ar
solutions of equation (4), then two important relations are true:

$$(\Phi_z\chi)_{\bar z} = (\Phi_{\bar z}\chi)_z, \quad (\Phi_z\chi - \Phi\chi_z)_{\bar z} = (\Phi\chi_{\bar z} - \Phi_{\bar z}\chi)_z.$$

It follows from the second of them, that the integral

$$\omega(\Phi,\ \chi) = \int_{(z_0,\ \bar z_0)}^{(z,\ \bar z)} (\Phi_z\chi - \chi_z\Phi)dz + (\chi_{\bar z}\Phi - \chi\Phi_{\bar z})d\bar z,$$

does not depend on the integration path.

b). We shall proceed to the analysis of the equation (5) from the generalized Dar
Transformation point of view. Let $\Psi,\ \Psi_1$ are two linearly independent solutions of (5). Foll
[1, 3], we assume that

$$\tilde\Psi(z,\bar z) = \frac{\Omega(\Psi,\Psi_1)}{\ldots}$$

where $\Omega = \Omega(\Psi, \Psi_1) = \Omega(z, \bar{z})$ is the functional, given on direct product of two copies of s of "wave functions ". After substitution of the anzats (8) in equation (5), and requirin covariance of this equation under transformation $\Psi \to \tilde{\Psi}$, $U \to \tilde{U}$, we obtain the equ which do not contain already the potential:

$$\Omega_{z\bar{z}} - (\ln \Psi_1)_{\bar{z}}\Omega_z - (\ln \Psi_1)_z \Omega_{\bar{z}} + q_1 \Omega = 0.$$

Here $q_1 = q_1(z, \bar{z})$ is a complex-valued function ("parameter" of separation of variables), duced in order to obtain the real potential $\tilde{V}(z, \bar{z})$ for the case of $V(z, \bar{z}) = \bar{V}(z, \bar{z})$ (conser medium). If $V(z, \bar{z}) \neq \bar{V}(z, \bar{z})$ and $\tilde{V}(z, \bar{z}) \neq \bar{\tilde{V}}(z, \bar{z})$ (nonconservative medium), one may $q_1 = 0$ in (9).

Equation (9) allows the physical interpretation: it looks like the two-dimensional stati Schrodinger equation for charged particle in a non-homogenious stationary electromagneti [1]. Actually, this equation can be written as (see, for example, [10]):

$$\triangle\psi - 2i(\mathbf{A}\nabla\psi) - (\mathbf{A}^2 + \phi - E)\psi = 0,$$

where $\mathbf{A}(x, y) = (A_1, A_2)$ is the vector potential, with the gauge condition $div\,\mathbf{A} = 0$, $\phi = \phi$ is the scalar potential, E is the energy value, and the system of units $\hbar = c = e = 1$, m was used (m is the mass of the particle). In variables $z, \ \bar{z}$:

$$\psi_{z\bar{z}} - \frac{1}{4}(iA_1 - A_2)\psi_z - \frac{1}{4}(iA_1 + A_2)\psi_{\bar{z}} - \frac{1}{4}[\mathbf{A}^2 + \phi - E]\psi = 0.$$

Comparing (11) and (9), we see, that analogues of coefficients at Ω_z, $\Omega_{\bar{z}}$ are expressed as combinations of the components of the vector-potential, and the analogue of function q terms of the square of its module and of the scalar potential.

Besides, the separation of variables leads to the following dressing relation for potential number):

$$\tilde{V}(z, \bar{z}) = V(z, \bar{z}) - 8(\ln \Psi_1)_{z\bar{z}} - 4q_1(z, \bar{z}).$$

In the conservative case the requirement of reality reads:

$$(arg\,\Psi_1)_{z\bar{z}} = -4\,\mathrm{Im}\,q_1.$$

For the double dressing procedure, with

$$\tilde{\tilde{\Psi}}(z, \ \bar{z}) = \frac{\Omega(\Psi_2[1], \ \Psi[1])}{\Psi_2[1]},$$

and taking into account (12), we obtain

$$\tilde{\tilde{V}}(z, \ \bar{z}) = V - 8(\ln \Omega(\Psi_1, \ \Psi_2))_{z\,\bar{z}} - 4\{q_1 + q_2[1]\},$$

where $\Psi_2[1]$ is the fixed solution of equation (5) with $\Psi \to \Psi[1]$, $V \to V[1]$, and Ψ_2 $\Omega(\Psi_1, \ \Psi_2)/\Psi_1$, Ψ_2 is some fixed solution of equation (5).

It is clear, that similar to the case of standard DT, the dressing procedure propose can be multiply iterated. Setting $\Psi = \Psi[0]$, $\tilde{\Psi} = \Psi[1], \ldots, V = V[0]$, $\tilde{V} = V[1], \ldots$, we ha chains: $\Psi[0] \to \Psi[1] \to \ldots \to \Psi[N] \to \ldots$, and $V[0] \to V[1] \to \ldots \to V[N] \to \ldots$. In parti for the N-fold iteration of the dressing procedure, one will obtain the relations, express terms of the functions $\Omega(\Psi_i, \ \Psi_j)$ only [1].

c). Now some simplest examples of realization of the proposed approach will be consid

1. Let us take $V(z, \bar{z}) = 0$ in (5). Then $\Psi_j(z,\, \bar{z}) = \Psi_j^{(1)}(z) + \Psi_j^{(2)}(\bar{z})$, $\Psi_j^{(1)}$, $\Psi_j^{(2)}$ are ab functions, $j = 1,\, 2, \ldots, N, \ldots$. The relation (12) takes a form:

$$V[1] = -8[\ln\left(\Psi_1^{(1)}(z) + \Psi_1^{(2)}(\bar{z})\right)]_{z\bar{z}} - 4q_1.$$

In the case $V[1] = \bar{V}[1]$ the following condition

$$\mathrm{I}m\, q_1 = i\left[\frac{\Psi_z^{(1)}\Psi_{\bar{z}}^{(2)}}{(\Psi^{(1)} + \Psi^{(2)})^2} - \frac{\bar{\Psi}_{\bar{z}}^{(1)}\bar{\Psi}_z^{(2)}}{(\bar{\Psi}^{(1)} + \bar{\Psi}^{(2)})^2}\right]$$

must be satisfied. On the next step we have:

$$V[2] = V - 8(\ln\Omega(\Psi_1,\, \Psi_2))_{z\bar{z}} - 4[q_1 + q_2[1]),$$

and the functional $\Omega(\Psi_1, \Psi_2)$ satisfies the equation (9), if to replace $\Psi \to \Psi_1, \Psi_1 \to \Psi_2, q_1 -$ in it. It is not difficult to check, that for $q_1 = q_2[1] = 0$ this equation is solved by the fur $\Omega(\Psi_1,\, \Psi_2) = \omega(\Psi_1,\, \Psi_2)$, where the expression for $\omega(\Psi_1,\, \Psi_2)$ is defined in (7). After the s calculations we obtain:

$$\Omega(\Psi_1,\, \Psi_2) = \gamma_0 + 2(\Psi_1^{(1)}\Psi_2^{(2)} - \Psi_2^{(1)}\Psi_1^{(2)}) +$$

$$+\int_{(z_0,\, \bar{z}_0)}^{(z,\, \bar{z})} (\Psi_{1z}^{(1)}\Psi_2^{(1)} - \Psi_{2z}^{(1)}\Psi_1^{(1)})dz + (\Psi_{2\bar{z}}^{(2)}\Psi_1^{(2)} - \Psi_{1\bar{z}}^{(2)}\Psi_2^{(2)})d\bar{z},$$

where γ_0 is a complex constant. This expression, without using of the equation (9), was obt for the first time in [1].

2. Let's assume in (4) that $V = V(x, y) = 1$ at $-\infty < x < +\infty$, $y > 0$. Applying the F transformation in the variable x, we shall construct the solution Ψ_j ($j = 1, 2, \ldots, N, \ldots$), is bounded in x and goes to zero at $y \to \infty$:

$$\Psi_j(x,\, y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ikx - \sqrt{k^2+1}y} A_j(k).$$

Here $A_j(k)$ are some functions (functional parameters), such that $A_j(k) \in \mathbb{L}_2(-\infty, \infty$ the real case one obtains from (19) $A_j(k) = \bar{A}_j(-k)$ (in case of complex k we assume $\mathrm{Re}\,\sqrt{k^2+1} > 0$). Then

$$V[1] = 1 - 2\triangle\ln\{\int_{-\infty}^{+\infty} dk\, e^{ikx - \sqrt{k^2+1}y} A_1(k)\}.$$

The analogue of the second relation in (6) yields:

$$(\Psi_x\Psi_1 - \Psi_{1x}\Psi_x)_x = (\Psi_{1y}\Psi - \Psi_1\Psi_y)_y.$$

Therefore the solution $\Psi[1]$ is determined by a relation (8) at $\Omega(\Psi,\, \Psi_1) = \omega(\Psi,\, \Psi_1)$, whe

$$\omega(\Psi, \Psi_1) = \int_{(x_0,\, y_0)}^{(x,y)} (\Psi_{1y}\Psi - \Psi_1\Psi_y)dx + (\Psi_x\Psi_1 - \Psi_{1x}\Psi)dy.$$

On the next step we obtain:

In the particular case of $A_j(k) = c_j\delta(k-p_j) + d_j\delta(k-r_j)$, with $\delta(.)-$ the Dirac delta- fun $c_j,\ d_j,\ p_j,\ r_j \in \mathbb{R}-$ constants, $j = 1, 2, \ldots, N, \ldots$, taking (due to linearity of the equatio the real part of the solution Ψ_j we shall have:

$$\Psi_j(x,\ y) = c_j e^{-\sqrt{p_j^2+1}y}\cos p_j x + d_j e^{-\sqrt{r_j^2+1}y}\cos r_j x.$$

Thus, the dressing formula (15) gives:

$$V[1] = 1 - 2\triangle \ln\left(c_1 e^{-\sqrt{p_1^2+1}y}\cos p_1 x + d_1 e^{-\sqrt{r_1^2+1}y}\cos r_1 x\right).$$

Completely similarly, it is possible to obtain from (15), (24) the explicit representation for though rather cumbersome.

3. Let us consider a case, when the initial solution is determined by the Coulomb pote i.e. in (4) we assume that $V(x,\ y) = (\sqrt{x^2+y^2}\,)^{-1}$, $x \in (-\infty,\ \infty)$, $y \in (-\infty,\ \infty)$. S this equation in the polar coordinates and coming back to the cartesian ones, (with require that the solution is finite at $x = y = 0$), we obtain [11]:

$$\Psi_m(x,y) \sim e^{im\arctan\frac{y}{x}} J_{2i\sqrt{m}}\left(2i(x^2+y^2)^{\frac{1}{4}}\right),\quad m = 1,\ 2, \ldots,\ N, \ldots,$$

where $J_\nu(.)$ is the Bessel function. Then for real potential we find:

$$V[1] = \frac{1}{\sqrt{x^2+y^2}} - 2\triangle \ln\left\{\mathrm{Re}\left[\,e^{im\arctan\frac{y}{x}} J_{2i\sqrt{m}}\left(2i(x^2+y^2)^{\frac{1}{4}}\right)\right]\right\}.$$

It is clear now, that already on a first step we obtain the potential which is not ame for separation of variables. Hence, all subsequent functions of the chain $\Psi[k]$, $V[k]$ obe property as well.

Potentials of the form (26) and (28), and of the corresponding chains, are integrable, i. equations, produced by them, have the exact solutions.

In two following examples we consider a case $q_1 \neq 0$ when the equation (9) allows the sin solutions.

4. Let $\Omega_z = 0$, i.e. Ω is an antiholomorphic function in some domain $\mathbb{D} \subset \mathbb{C}$. From (have:

$$(\ln\Omega)_{\bar{z}} = \frac{q_1}{(\ln\Psi_1)_z} \quad \text{at the condition, that} \quad \left(\frac{q_1}{(\ln\Psi_1)_z}\right)_z = 0.$$

To solve this equation we use the formula of $\bar{\partial}$-problem (see, for example, [6]). Assu that

$$\Omega_{|\partial\mathbb{D}} = C_0,$$

where C_0 is some complex constant, we find:

$$\ln\Omega(\bar{z}) = C_0 + \frac{1}{2\pi i}\int\int_{\mathbb{D}} \frac{q_1}{(\ln\Psi_1)_\zeta(\zeta - z)}\,d\zeta\wedge d\bar{\zeta}.$$

Here $\zeta = \zeta_R + i\zeta_I$, $d\zeta\wedge d\bar{\zeta} = -2i d\zeta_R\, d\zeta_I$. The correctness of this expression follows fro well-known relation of the theory of generalized functions ($\bar{\partial} \equiv \partial_{\bar{z}}$):

$$\bar{\partial}\left(\frac{1}{\pi(z-z_0)}\right) = \delta(z - z_0).$$

5. Analogously, let $\Omega_{\bar z} = 0$, i.e. Ω is an holomorphic function in $\mathbb{D} \subset \mathbb{C}$. From (9) we ol

$$(\ln \Omega)_z = \frac{q_1}{(\ln \Psi_1)_{\bar z}} \quad \text{at the condition, that} \quad (\frac{q_1}{(\ln \Psi_1)_{\bar z}})_{\bar z} = 0.$$

Assuming, that

$$\Omega_{|\partial \mathbb{D}} = C_1,$$

where C_1 is complex constant, from (33) we have

$$\ln \Omega(z) = C_1 + \frac{1}{2\pi i} \int\int_{\mathbb{D}} \frac{q_1}{(\ln \Psi_1)_{\bar\zeta}(\bar\zeta - \bar z)}\, d\zeta \wedge d\bar\zeta,$$

d). Now we shall find the quantum-mechanical sense of the transformations (8). Fo purpose we shall consider more general, than (5), equation:

$$\Psi_{z\bar z} = \frac{1}{4}(V(z, \bar z) - \lambda)\Psi,$$

where $\lambda \in \mathbb{C}$ is a complex parameter. It is possible also to use here the transformation (8 dressing relation (15). Let us assume that $V(z, \bar z) = z\bar z$, corresponding to the two-dimens isotropic harmonic oscillator. Then the equation (36) can be rewritten as

$$-4\Psi_{z\,\bar z}(z,\, \bar z,\, \lambda) + z\bar z\Psi(z,\, \bar z,\, \lambda) = \lambda\Psi(z,\, \bar z,\, \lambda).$$

For $\lambda = \lambda_1 = -2$ this equation has the solution $\Psi_1(z,\, \bar z) = \exp(\frac{1}{2} z\bar z)$, and agrees with $\tilde V(z,\, \bar z) = z\bar z - 4 - 4q_1$. Thus we obtain the equation

$$-4\Psi_{z\bar z}[1](z,\, \bar z,\, \lambda) + (z\bar z - 4q_1)\Psi[1](z,\, \bar z,\, \lambda) = (\lambda + 4)\Psi[1](z,\, \bar z,\, \lambda).$$

Comparing (37) and (38) for $q_1 = 0$, we find

$$\Psi[1](z,\, \bar z,\, \lambda) = \Psi(z,\, \bar z,\, \lambda + 4),$$

i.e. the transformation (8), together with the requirement of covariance, acts as the quan mechanical creation operator of particles [2], and the quantity $-4q_1$ can be interpreted a energy level shift (in a complex case). In this sense we have two-dimensional generalizati the standard DT [1].

Here the equation for functional Ω follows from (9):

$$\Omega_{xx} + \Omega_{yy} - 2x\,\Omega_x - 2y\,\Omega_y + q_1\Omega = 0.$$

This equation, which can be called the generalized Helmholtz equation, is known in the m matical literature [12]. It arises under consideration of orthogonal polynomials in two ind dent variables. For $q_1 = 2(n + m)$ it has the solution

$$\Omega(x,\, y) = \mathcal{F}_{n+m,m}(x,\, y) = H_n(x)H_m(y),$$

where $H_l(.)$ are polynomials of Chebychev-Ermit, $m, n = 0, 1, \ldots$. Here the generalized con of orthogonality is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,\, y)\mathcal{F}_{nm}(x,\, y)\mathcal{F}_{ks}(x,\, y)\, dx\, dy = \delta_{n-m,k-s}\,\delta_{ms},$$

where $h(x,\ y) = \exp\{-(x^2+y^2)\}$ is the weight function, and δ_{ij} is the Kronecker symbol

For $q_1 = 0$ the solution of equation (40) also can be factorized. Setting $\Omega(x, y) = \Omega_1(x))\Omega$ we have:

$$\Omega_{1xx} - 2x\Omega_{1x} - \beta\Omega_1 = 0, \qquad \Omega_{2yy} - 2y\Omega_{2y} + \beta\Omega_2 = 0,$$

where β is the "separation of variables parameter". For $\beta = 2n$, where $n = 1,\ 2, \ldots,$ the s of these equations has the solution in terms of the Chebychev-Hermit polynomials: Ω_2 $H_n(y)$. Setting $\Omega_1(x) \equiv \mathcal{P}(x) = \sum_{k=0}^{\infty} a_k x^k$, from the first of the equations (43) we fin reccurent relation:

$$a_{k+2} = \frac{2(k+n)}{(k+2)(k+1)} a_k,$$

where the numbers $a_0,\ a_1$ remain arbitrary, and the power series for function $\mathcal{P}(x)$ conv uniformly on the entire axis.

Thus, in the dressing procedure the solution of equation (36) is

$$\tilde{\Psi}(x,\ y) = \Psi[1](x,\ y) = c_n e^{x^2+y^2}\mathcal{P}(x)H_n(y),$$

where c_n is a constant. Besides that, similarly to the one-dimensional case (one-dimens stationary Schrodinger equation), the states described by the relation (45) are not normali The detailed investigation of the creation and annihilation operators in two and more s dimensions was performed in the monograph [10].

3. A method of $\bar{\partial}$ - problem.

In this Section we consider the application of the $\bar{\partial}$-problem method for the simplest c diffraction on half-planes (see, for example, [14]):

$$\triangle u + k^2 u = 0,$$

and [3]

$$u = -u^{(i)}(x,\ y) = -e^{-ikx\cos\theta} \ \text{ at } \ x < 0,\ \ y \to \pm 0,$$

$$u = O(\frac{1}{\sqrt{r}}), \quad \frac{\partial u}{\partial r} - iku = o(\frac{1}{\sqrt{r}}) \ \text{ at } \ r = \sqrt{x^2+y^2} \to \infty.$$

Here $u = u(x,\ y) = u^{(tot)}(x,\ y) - u^{(i)}(x,\ y)$, $u^{(tot)}$ is the total potential, $u^{(i)})$ is the pot of a falling wave, $u^{(i)} = \exp(-ikx\cos\theta - iky\sin\theta)$, $\triangle$ is the two-dimensional Laplace ope $k = k_1 + ik_2,\ k_1,\ k_2 > 0,\ 0 \le \theta \le \pi$.

Let us introduce the variables z and $\bar{z}$, and let us assume $u(z,\ \bar{z}) = v(z\,\bar{z})\exp(k_- z +$ where $k_+ = -(i/2)k\exp(i\theta)$, $k_- = -(i/2)k\exp(-i\theta)$. Then in the z - representation the pro (46)-(48) can be written in terms of function $v = v(z,\ \bar{z}) \equiv v(z,\ \bar{z},\ k_+,\ k_-)$ as

$$v_{z\bar{z}} + k_+ v_z + k_-\ v_{\bar{z}} = 0,$$

$$v(z,\ \bar{z}) = -1 \ \text{ at } \ \ z + \bar{z} < 0,\ z - \bar{z} \to \pm i0.$$

The radiation condition (48) gives

$$v(z,\ \bar z) = o(1)\ \text{ at } |z| \to \infty.$$

It is not difficult to obtain the symmetry property from the equation (49):

$$v(z,\ \bar z;\ k_+\ k_-) = v(\bar z,\ z;\ k_-,\ k_+).$$

In order to derive the integral equations we note, that for any complex-valued fun $g(z, \bar z) \in C(\bar{\mathbb{D}}) \bigcap C^1(\mathbb{D})$, $\mathbb{D} \subset \mathbb{C}$ the Green formulas [13] can be used:

$$\oint_{\partial\mathbb{D}} g d\zeta = -\int_{\mathbb{D}}\int g_{\bar\zeta}\, d\zeta \wedge \bar\zeta,\quad \oint_{\partial\mathbb{D}} g d\bar\zeta = \int_{\mathbb{D}}\int g_\zeta\, d\zeta \wedge d\bar\zeta.$$

Setting $g(\zeta, \bar\zeta, l_1, l_2) = g_1(\zeta, \bar\zeta, l_1, l_2) = w(\zeta,\ \bar\zeta,\ l_1,\ l_2)/(\zeta - z)$, $g(\zeta, \bar\zeta, l_1, l_2) = g_2(\zeta, \bar\zeta, l_1,$ $w(\zeta,\ \bar\zeta,\ l_1,\ l_2)/(\bar\zeta - \bar z)$, $w(\zeta,\ \bar\zeta) \equiv w(\zeta,\ \bar\zeta,\ l_1,\ l_2) \in C(\bar{\mathbb{D}}) \bigcap C^1(\mathbb{D})$, where $l_1,\ l_2 \in \mathbb{C}$ are param applying to these functions the Cauchy's formula with paths $\partial\mathbb{D} \bigcup |\zeta - z| = \epsilon_1$, and $\partial\mathbb{D} \bigcup |\bar\zeta$ ϵ_2 accordingly, where $\epsilon_1,\ \epsilon_2 > 0$, and passing to limits $\lim \epsilon_{1,2} = 0$, we obtain the in representations:

$$w(z,\ \bar z) = \int_{\partial\mathbb{D}} \frac{d\zeta}{2\pi i}\ \frac{w(\zeta,\ \bar\zeta)}{\zeta - z} + \int_{\mathbb{D}}\int \frac{d\zeta \wedge d\bar\zeta}{2\pi i}\ \frac{\partial_{\bar\zeta} w(\zeta,\ \bar\zeta)}{\zeta - z},$$

$$w(z,\ \bar z) = -\int_{\partial\mathbb{D}} \frac{d\bar\zeta}{2\pi i}\ \frac{w(\zeta,\ \bar\zeta)}{\bar\zeta - \bar z} + \int_{\mathbb{D}}\int \frac{d\zeta \wedge d\bar\zeta}{2\pi i}\ \frac{\partial_\zeta w(\zeta,\ \bar\zeta)}{\bar\zeta - \bar z}.$$

To apply these representations, equation (49) can be rewritten as

$$\partial_{\bar z}(\partial_z + k_-)v(z,\ \bar z) = -k_+\partial_z v(z,\ \bar z),\quad \partial_z(\partial_{\bar z} + k_+)v(z,\ \bar z) = -k_-\partial_{\bar z} v(z,\ \bar z).$$

We assume, that

$$v_z(z,\ \bar z) = o(1),\ \ v_{\bar z}(z,\ \bar z) = o(1)\ \text{ at } |z| \to \infty.$$

Choosing the path as $\partial\mathbb{D} = \Gamma \bigcup C_R$, where C_R is the circle of radius R, and contour Γ co of the straight line $y = \pm i0$, at $x < 0$, and the half-circle of small radius at $x \geq 0$, we o from (54)-(56) :

$$v_z(z,\ \bar z) + k_- v(z,\ \bar z) = \int_{\Gamma \bigcup C_R} \frac{d\zeta}{2\pi i}\ \frac{v_\zeta(\zeta,\ \bar\zeta) + k_- v(\zeta,\ \bar\zeta)}{\zeta - z} - \int_{\mathbb{D}}\int \frac{d\zeta \wedge d\bar\zeta}{2\pi i}\ \frac{k_+ v_\zeta(\zeta,\ \bar\zeta)}{\zeta - z},$$

$$v_{\bar z}(z,\ \bar z) + k_+ v(z,\ \bar z) = -\int_{\Gamma \bigcup C_R} \frac{d\bar\zeta}{2\pi i}\ \frac{v_{\bar\zeta}(\zeta,\ \bar\zeta) + k_+ v(\zeta,\ \bar\zeta)}{\bar\zeta - \bar z} - \int_{\mathbb{D}}\int \frac{d\zeta \wedge d\bar\zeta}{2\pi i}\ \frac{k_- v_{\bar\zeta}(\zeta,\ \bar\zeta)}{\bar\zeta - \bar z}.$$

These relations can be also simplified. Indeed, taking into account (51) and (57) we have

$$\lim_{R\to\infty} \int_{C_R} \frac{d\zeta}{2\pi i}\ \frac{v_\zeta(\zeta,\ \bar\zeta) + k_- v(\zeta,\ \bar\zeta)}{\zeta - z} = \lim_{R\to\infty} \int_{C_R} \frac{d\bar\zeta}{2\pi i}\ \frac{v_{\bar\zeta}(\zeta,\ \bar\zeta) + k_+ v(\zeta,\ \bar\zeta)}{\bar\zeta - \bar z} = 0.$$

Besides that, condition (50) gives:

$$\int_{\Gamma} \frac{d\bar{\zeta}}{2\pi i} \frac{v_{\bar{\zeta}}(\zeta,\, \bar{\zeta}) + k_{+}(\zeta,\, \bar{\zeta})}{\bar{\zeta} - \bar{z}} = \int_{-\infty}^{0} \frac{d\xi}{2\pi i} \frac{v_{\bar{\zeta}}^{+}(\xi) - v_{\bar{\zeta}}^{-}(\xi)}{\xi - \bar{z}}.$$

Then equations (58)-(59) yield:

$$v_{z}(z,\, \bar{z}) + k_{-} v(z,\, \bar{z}) = \int_{-\infty}^{0} \frac{d\xi}{2\pi i} \frac{v_{\zeta}^{+}(\xi) - v_{\zeta}^{-}(\xi)}{\xi - z} - k_{+} \int_{\mathbb{D}}\!\!\int \frac{d\zeta \wedge d\bar{\zeta}}{2\pi i} \frac{v_{\zeta}(\zeta,\, \bar{\zeta})}{\zeta - z},$$

$$v_{\bar{z}}(z,\, \bar{z}) + k_{+} v(z,\, \bar{z}) = -\int_{-\infty}^{0} \frac{d\xi}{2\pi i} \frac{v_{\bar{\zeta}}^{+}(\xi) - v_{\bar{\zeta}}^{-}(\xi)}{\xi - \bar{z}} - k_{-} \int_{\mathbb{D}}\!\!\int \frac{d\zeta \wedge d\bar{\zeta}}{2\pi i} \frac{v_{\bar{\zeta}}(\zeta,\, \bar{\zeta})}{\bar{\zeta} - \bar{z}},$$

where we used notations:

$$v_{\zeta}^{\pm}(\xi) = \frac{1}{2}(\partial_{\xi} - i\partial_{\eta}) v(\xi + i\eta,\, \xi - i\eta)_{|\eta \to \pm 0}, \quad v_{\bar{\zeta}}^{\pm}(\xi) = \frac{1}{2}(\partial_{\xi} + i\partial_{\eta}) v(\xi + i\eta,\, \xi - i\eta)_{|\eta \to \pm 0}.$$

The equations (63)-(64) allow, in particular, to obtain the system of equations which con
a derivative of a field on a negative half-axis with derivative in the domain $\mathbb{D}$. Reall
$z = x < 0$ and taking into account the boundary condition (50), equations (63) can be w
as

$$v_{z}^{+}(x) - k_{-} = \int_{-\infty}^{0} \frac{d\xi_1}{2\pi i} \frac{v_{\zeta_1}^{+}(\xi_1) - v_{\zeta_1}^{-}(\xi_1)}{\xi_1 - x - i0} - k_{+} \int_{\mathbb{D}}\!\!\int \frac{d\zeta_1 \wedge d\bar{\zeta}_1}{2\pi i} \frac{v_{\zeta_1}(\zeta_1,\, \bar{\zeta}_1)}{\zeta_1 - x},$$

$$v_{z}^{-}(x) - k_{-} = \int_{-\infty}^{0} \frac{d\xi_1}{2\pi i} \frac{v_{\zeta_1}^{+}(\xi_1) - v_{\zeta_1}^{-}(\xi_1)}{\xi_1 - x + i0} - k_{+} \int_{\mathbb{D}}\!\!\int \frac{d\zeta_1 \wedge d\bar{\zeta}_1}{2\pi i} \frac{v_{\zeta_1}(\zeta_1,\, \bar{\zeta}_1)}{\zeta_1 - x},$$

and equation (64) as

$$v_{\bar{z}}^{+}(x) - k_{+} = -\int_{-\infty}^{0} \frac{d\xi_2}{2\pi i} \frac{v_{\bar{\zeta}_2}^{+}(\xi_2) - v_{\bar{\zeta}_2}^{-}(\xi_2)}{\xi_2 - x + i0} - k_{-} \int_{\mathbb{D}}\!\!\int \frac{d\zeta_2 \wedge d\bar{\zeta}_2}{2\pi i} \frac{v_{\bar{\zeta}_2}(\zeta_2,\, \bar{\zeta}_2)}{\bar{\zeta}_2 - x},$$

$$v_{\bar{z}}^{-}(x) - k_{+} = -\int_{-\infty}^{0} \frac{d\xi_2}{2\pi i} \frac{v_{\bar{\zeta}_2}^{-}(\xi_2) - v_{\bar{\zeta}_2}^{-}(\xi_2)}{\xi_2 - x - i0} - k_{-} \int_{\mathbb{D}}\!\!\int \frac{d\zeta_2 \wedge d\bar{\zeta}_2}{2\pi i} \frac{v_{\bar{\zeta}_2}(\zeta_2,\, \bar{\zeta}_2)}{\bar{\zeta}_2 - x}.$$

The system of equations (66)-(69) is closed system of singular integral equations on th
x, to find the functions v_z^{+}, v_z^{-} and $v_{\bar{z}}^{+}$, $v_{\bar{z}}^{-}$, respectively, from the values of v_z and $v_{\bar{z}}$
arbitrary point of domain $\mathbb{D}$.

These functions can be also calculated using another approach. Indeed, equation (49
be rewritten as

$$\left(\frac{1}{2}\,\partial_{\bar{z}} + k_{+}\right) v_{z}(z,\, \bar{z}) = -\left(\frac{1}{2}\,\partial_{z} + k_{-}\right) v_{\bar{z}}(z,\, \bar{z}).$$

Then we construct the generalized function - fundamental solution of the operator ((1/4)
$i\partial_y) + k_{+}$), which satisfies the equation

$$[(\partial_x + i\partial_y) + 4k_{+}] G_1(x,\, y,\, k_{+}) = 4\delta(x)\delta(y).$$

Thus from the equation (70) it follows:

Similarly we can find the generalized function - fundamental solution of the operator ((1/
$i\partial_y) + k_-$):

$$[(\partial_x - i\partial_y) + 4k_-]G_2(x,\, y,\, k_-) = 4\delta(x)\delta(y).$$

From (70) and (73) we obtain:

$$v_{\bar{z}}(x,\, y) = -\int_{\mathbb{D}}\int d\xi\, d\eta\, G_2(x-\xi,\, y-\eta,\, k_-)[\partial_\xi + i\partial_\eta + 4k_+]v_\zeta(\xi,\, \eta).$$

The solutions G_1 and G_2 are connected among themselves by the symmetry relation:

$$G_1(z,\, \bar{z},\, k_+) = -\bar{G}_2(-z,\, \bar{z},\, -\bar{k}_+), \quad G_2(z,\, \bar{z},\, -k_-) = -\bar{G}_1(-z,\, -\bar{z},\, -\bar{k}_-).$$

The system of equations (72), (74) is a system of integral equations of the convolution
and it can be solved by the Fourier transformation (for generalized functions). Such calcula
as well as the explicit expressions for functions G_1, G_2 and the comparison with the resu
[14] will be given later.

The approach developed above in this Section is seems to be an alternative to the tradi
Wiener-Hopf approach. It would be interesting to generalize this method for the case of
dimensional problem of diffraction.

Acknowledgements

The author is grateful to M.V.Ioffe for attention and A.V.Tukhtin for useful discussi
questions of the theory of diffraction.

References

[1] Matveev V.B. & Salle M.A, 1991, Darboux Transformation and Solitons. Berlin. Spr
Verlag.

[2] Sabatier P., 1998, *On multidimensional Darboux transformation, Inv. Prob,* **Vol.1**
355-366.

[3] Gutshabash E.Sh. & Salle M.A., 2001, *Darboux-type anzats in nonstationary pro
of Quantum Mechanics and Kinetics.* In Proceedings of International Seminar " D
Difraction-2001 ", May 29-31, 2001, St. Petersburg, Russia, pp.116-127.

[4] Athorne C. & Nimmo J., 1991, *On the Moutard transformation for integrable partial
ential Equations, Inv. Prob.,* **Vol. 7,** pp. 809-826.

[5] Lipovskii V.D., 1989, *Private Communication.*

[6] Ablovitz M. & Clarcson P., 1991, *Solitons, Nonlinear Evolution Equations and In
Scattering,* Cambridge.

[7] Konopelchenko B., 1993, *Solitons in Multidimensions,* Singapore.

[9] Novikov S.P. & Veselov A.P., 1984, *D*okl. Akad. Nauk SSSR, **Vol. 279 (1)**.

[10] Messia A., 1978, *Quantum Mechanics,* **vol.1**, Moscow, Nauka (in Russian).

[11] Kamke E., 1961, *Spravochnik po Obyknovennym differential'nym uravneniam. Mosko* Russian).

[12] Suetin P.K., 1988, *Ortogonal'nye Mnogochleny po dvum peremennym,* Moskow, Nau Russian).

[13] Vekua I.N., 1988, *Obobschennye Analiticheskie funksii.* Moskow, Nauka (in Russian).

[14] Noble B., 1958, *Methods based on the Wiener-Hopf techique for the solution of p differential Equations.* Pergamon Press.